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Anisotropic Scattering in Accordance with Pomraning Phase Function

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The emergent intensity $I(0, \mu)$ from the equation of transfer in anisotropically scattering medium with Pomraning phase function is derived in n^{th} approximation by using Chandrasekhar's discrete ordinate method.

KEY WORDS: discrete ordinate method; pomraning phase function; characteristic equation; emergent intensity.

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1. INTRODUCTION

Pomraning (1998) introduced a new phase function where the radiation is scattered according to the Rayleigh like phase function. Viik (2001) used it to derive the intensities in a homogeneous plane-parallel optically semi-infinite atmosphere where there are sources of radiation infinitely deep in the atmosphere and where the radiation is scattered according to the Rayleigh like Pomranning phase function. He considered the accuracy of the phase function on the basis of the Milne problem in a homogeneous plane parallel atmosphere by solving the vector transfer equation using the Chandrasekhar's discrete ordinate method and the respective scalar equations by using Chandrasehar-Ivanov principles of invariance to reduce the boundary value problem into a Cauchy initial-value problem. By using the same phase function, Viik and McCormick (2002) performed approximate polarized Rayleigh transfer calculations with a scalar radiative transfer equation.

Ghosh, Mukherjee and Karanjai (2004) studied some approximate form of H-function already studied by Karanjai (1968) and Karanjai and Sen (1971) to

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the case of anisotropically scattering atmosphere with Pomraning Phase function. Islam, Mukherjee, and Karanjai (2004) solved the equation of radiative transfer with Pomraning Phase function and a non-linear source in a plane semi-infinite atmosphere with axial symmetry by Laplace Transform and Wiener-Hopf technique. They determined the emergent intensity in terms of Chandrasekhar's H-function and the intensity at any optical depth by inversion.

We have used this phase function here to derive the emergent intensity in n^{th} approximation by using the Chandrasekhar's discrete ordinate method.

2. THE RADIATIVE TRANSFER EQUATION AND THE BOUNDARY CONDITIONS

2.1. The Radiative Transfer Equation

The equation of radiative transfer appropriate for the problem is

$$\mu \frac{dI(\tau, \mu)}{d\tau} = I(\tau, \mu) - \frac{1}{2}\varpi_0 \int_{-1}^{+1} p(\mu, \mu')I(\tau, \mu')d\mu' \tag{1}$$

where I is the intensity, τ the optical depth measured from the upper boundary of the atmosphere, μ, the cosine of the angle between the direction of travel of a photon and the outward normal drawn at the surface of its incidence, ϖ_0 is the albedo of single scattering and $p(\mu, \mu')$ is the Pomraning Phase function (Viik, 2001; Pomraning, 1998), given by

$$p(\mu, \mu') = 1 + \frac{\lambda}{2}P_2(\mu)P_2(\mu') = 1 + \frac{\lambda}{8}(3\mu^2 - 1)(3\mu'^2 - 1)$$

$$\text{i.e. } p(\mu, \mu') = 1 + \frac{\lambda}{8}(9\mu^2\mu'^2 - 3\mu'^2 - 3\mu^2 + 1); \tag{2}$$

where $P_2(\mu)$ is the Legendre Polynomial of second order and λ is a constant given by

$$\lambda = \frac{5}{5 - 3\varpi_0} \tag{3}$$

Using the Eq. (2), the Eq. (1) can be put in the form:

$$\mu \frac{dI(\tau, \mu)}{d\tau} = I(\tau, \mu) - \frac{1}{2}\varpi_0 \int_{-1}^{+1} \left\{1 + \frac{\lambda}{8}(9\mu^2\mu'^2 - 3\mu'^2 - 3\mu^2 - 1)\right\} I(\tau, \mu')d\mu' \tag{4}$$

2.2. Boundary Conditions for solving the Transfer Equation

The equation of transfer (4) is to be solved subject to the boundary conditions:

(i)

$$I(0,\mu)=0 \qquad \text{for } -1<\mu\leq 0 \tag{5}$$

(ii)

$$I(\tau,\mu)e^{-\tau}\rightarrow 0 \qquad as\ \tau\rightarrow\infty \tag{6}$$

3. SOLUTION OF THE EQUATION

3.1. Solution of the Equation in n^{th} Approximation

Now, following Chandrasekhar (1960), we form the following set of discrete equations:

$$\mu_i\frac{dI_i}{d\tau}=I_i-\frac{1}{2}\varpi_0\sum_j\left\{1+\frac{\lambda}{8}\left(9\mu_i^2\mu_j^2-3\mu_j^2-3\mu_i^2+1\right)\right\}I_ja_j; \qquad i=\pm1,\pm2,\ldots,\pm n \tag{7}$$

where j runs from $-n$ to $+n$ except the point $j=0$, $\mu_{-i}=-\mu_i$, $a_{-j}=a_j$ and $I_i=I(\tau,\mu_i)$.

Now the system of $2n$ Eq. (7) admits integrals of the form

$$I_i=g_ie^{-k\tau} \tag{8}$$

Then

$$(1+\mu_ik)g_i=\frac{\varpi_0}{16}\sum_j\left\{\left((8+\lambda)-3\lambda\mu_j^2\right)+3\lambda(3\mu_j^2-1)\mu_i^2\right\}a_jg_j \tag{9}$$

$$\Rightarrow g_i=\varpi_0\frac{\rho+\rho_1\mu_i^2}{1+\mu_ik}, \tag{10}$$

where ρ and ρ_1 are constants (independent of i).

Applying the Eq. (10) in the Eq. (9), we get

$$\begin{aligned}16\rho+16\rho_1\mu_i^2=&\{(8+\lambda)\rho\varpi_0D_0(k)+(-3\rho\lambda\varpi_0+(8+\lambda)\rho_1\varpi_0)D_2(k)\\&-3\rho_1\lambda\varpi_0D_4(k)\}+\{-3\rho\lambda\varpi_0D_0(k)+(-3\rho_1\lambda\varpi_0\\&+9\rho\lambda\varpi_0)D_2(k)+9\rho_1\lambda\varpi_0D_4(k)\}\mu_i^2\end{aligned} \tag{11}$$

where

$$D_\ell(x) = \sum_j \frac{a_j\ \mu_j^\ell}{1+\mu_j\ x} \tag{12}$$

which produces the following two equations:

$$(16-(8+\lambda)\varpi_0 D_0 + 3\lambda\varpi_0 D_2)\rho - ((8+\lambda)\varpi_0 D_2 - 3\lambda\varpi_0 D_4)\rho_1 = 0 \tag{13a}$$

and

$$(-9\lambda\varpi_0 D_2 + 3\lambda\varpi_0 D_0)\rho + (16+3\lambda\varpi_0 D_2 - 9\lambda\varpi_0 D_4)\rho_1 = 0 \tag{13b}$$

which, on elimination of ρ and ρ_1, will give

$$32 + 12\lambda\varpi_0 D_2 - 18\lambda\varpi_0 D_4 - 2(8+\lambda)\varpi_0 D_0 - 9\lambda\varpi_0^2\left(D_2^2 - D_0 D_4\right) = 0$$

So, by using the Eq. (36) of Appendix-A, we get

$$9\lambda(1+\varpi_0)\varpi_0 D_4 - 3\lambda\varpi_0(2+\varpi_0)D_2 + (8+\lambda)\varpi_0 D_0 = 16 \tag{14}$$

3.1.1 Characteristic Equation

From the Eq. (14)

$$\frac{9\lambda}{16}(1+\varpi_0)\varpi_0 D_4 - \frac{3\lambda}{16}\varpi_0(2+\varpi_0)D_2 + \frac{1}{16}(8+\lambda)\varpi_0 D_0 = 1$$

i.e.

$$\varpi_0 \sum_j \frac{a_j}{16(1+\mu_j\ k)}\left\{9\lambda(1+\varpi_0)\mu_j^4 - 3\lambda(2+\varpi_0)\mu_j^2 + (8+\alpha)\right\} = 1 \tag{15}$$

which is an equation in k of order $2n$, known as the *characteristic equation* and will give $2n$ non-zero distinct roots of the form $\pm k_\alpha$; $\alpha = 1, 2, \ldots, n$, if $\varpi_0 < 1$

3.1.2 Determination of $2n$ *independent integrals*

Now, from the first Eq. (13a)

$$\rho_1 = \frac{16-(8+\lambda)\varpi_0 D_0 + 3\lambda\varpi_0 D_2}{(8+\lambda)\varpi_0 D_2 - 3\lambda\varpi_0 D_4}\rho$$

Now we have

$$D_2 = -\frac{1}{k^2}(2-D_0)$$

and

$$D_4 = -\frac{2}{3k^2} - \frac{1}{k^4}(2-D_0)$$

Therefore,

$$\rho_1 = -\frac{(48k^4 - 18\lambda\varpi_0 k^2) - \{3(8+\lambda)\varpi_0 k^4 - 9\lambda\varpi_0 k^2\}D_0}{(48\varpi_0 k^2 - 18\lambda\varpi_0) - \{3(8+\lambda)\varpi_0 k^2 - 9\lambda\varpi_0\}D_0}\rho \tag{16}$$

But from the Eq. (14)

$$D_0 = \frac{16k^4 + 18\lambda(1+\varpi_0)\varpi_0 - 6\lambda\varpi_0 k^2}{9\lambda(1+\varpi_0)\varpi_0 - 3\lambda\varpi_0(2+\varpi_0)k^2 + (8+\lambda)\varpi_0 k^4} \tag{17}$$

Therefore, using the Eq. (17) in the Eq. (16), we get

$$\rho_1 = \frac{3\lambda(1+\varpi_0)\{3(1-\varpi_0) - k^2\}}{3\lambda\{(1+\varpi_0)^2 - 2\} + (8+\lambda)(1-\varpi_0)k^2}\,\rho \tag{18}$$

So, from the relation (10), we get

$$g_i = \varpi_0\rho\frac{3\lambda\{(1+\varpi_0)^2 - 2\} + (8+\lambda)(1-\varpi_0)k^2 + 3\lambda(1+\varpi_0)\{3(1-\varpi_0) - k^2\}\mu_i^2}{[3\lambda\{(1+\varpi_0)^2 - 2\} + (8+\lambda)(1-\varpi_0)k^2](1+\mu_i k)}$$

and therefore, the Eq. (7) admits $2n$ integrals of the form:

$$I_i = \varpi_0\rho\frac{3\lambda\{(1+\varpi_0)^2 - 2\} + (8+\lambda)(1-\varpi_0)k^2 + 3\lambda(1+\varpi_0)\{3(1-\varpi_0) - k^2\}\mu_i^2}{[3\lambda\{(1+\varpi_0)^2 - 2\} + (8+\lambda)(1-\varpi_0)k^2](1+\mu_i k)}\, e^{-k\tau} \tag{19}$$

and therefore, following Chandrasekhar (1960), the complete solution of the Eq. (7) can be put into the form:

$$I_i = \frac{3}{4}F \times \left\{\sum_{\lambda=1}^{n} \frac{3\lambda\{(1+\varpi_0)^2 - 2\} + (8+\lambda)(1-\varpi_0)k^2 + 3\lambda(1+\varpi_0)\{3(1-\varpi_0) - k^2\}\mu_i^2}{[3\lambda\{(1+\varpi_0)^2 - 2\} + (8+\lambda)(1-\varpi_0)k^2](1+\mu_i k)}\right\} \times L_\alpha\, e^{-k_\alpha\tau} \quad (i = 1, 2, \ldots, n) \tag{20}$$

where L_α are constants, obtainable by applying the boundary conditions (5) and F is the flux.

4. EMERGENT INTENSITY IN CLOSED FORM

4.1. Emergent Intensities in Terms of $S(\mu)$

Let us define the function $S(\mu)$ as follows:

$$S(\mu) = \sum_{\alpha=1}^{n} \frac{3\lambda\{(1+\varpi_0)^2 - 2\} + (8+\lambda)(1-\varpi_0)k^2 + 3\lambda(1+\varpi_0)\{3(1-\varpi_0) - k^2\}\mu^2}{[3\lambda\{(1+\varpi_0)^2 - 2\} + (8+\lambda)(1-\varpi_0)k^2](1-\mu\, k)}\, L_\alpha \quad (i = 1, 2, \ldots, n) \tag{21}$$

Now, from the Eq. (20), we get

$$I(0,\mu_i)=\frac{3}{4}F\left\{\sum_{\alpha=1}^{n}\frac{3\lambda\{(1+\varpi_0)^2-2\}+(8+\lambda)(1-\varpi_0)k^2+3\lambda(1+\varpi_0)\{3(1-\varpi_0)-k^2\}\mu_i^2}{[3\lambda\{(1+\varpi_0)^2-2\}+(8+\lambda)(1-\varpi_0)k^2](1+\mu_i k)}\right\}L_\alpha$$

$$(i=1,2,\ldots,n)$$

i.e.

$$I(0,\mu)=\frac{3}{4}F\left\{\sum_{\alpha=1}^{n}\frac{3\lambda\{(1+\varpi_0)^2-2\}+(8+\lambda)(1-\varpi_0)k^2+3\lambda(1+\varpi_0)\{3(1-\varpi_0)-k^2\}\mu^2}{[3\lambda\{(1+\varpi_0)^2-2\}+(8+\lambda)(1-\varpi_0)k^2](1+\mu k)}\right\}L_\alpha$$

$$(i=1,2,\ldots,n)$$

i.e.

$$I(0,\mu)=\frac{3}{4}\,FS(-\mu) \tag{22}$$

4.2. Emergent Intensities in Terms of $H(\mu)$

We observe, from the Eq. (22) that

$$I(0,\mu_i)=\frac{3}{4}\,FS(-\mu_i)$$

and from the boundary condition (5),

$$I(0,\mu_i)=0 \quad \text{for } -1<\mu_i\le 0$$

which produces

$$S(\mu_i)=0 \quad \text{for} -1<\mu_i\le 0 \tag{23}$$

which implies that μ_i's , where $i=1,\ 2,\ \ldots,\ n$, are the zeros of the polynomial $S(\mu)$.

So, the two polynomials $S(\mu)R(\mu)$ and $P(\mu)$, with

$$P(\mu)=\prod_{i=1}^{n}(\mu-\mu_i) \tag{24}$$

and

$$R(\mu)=\prod_{\alpha=1}^{n}(1-k_\alpha\mu) \tag{25}$$

have the zeros whose co-efficients of μ^n are 1 and $(-1)^n\,k_1\times k_2\ldots k_n$ respectively.

Therefore,

$$S(\mu) = (-1)^n k_1 k_2 \dots k_n \frac{P(\mu)}{R(\mu)} \tag{26}$$

which can also be expressed as

$$S(-\mu) = k_1 \cdot k_2 \dots k_n \mu_1 \cdot \mu_2 \dots \mu_n \cdot H(\mu)$$

where

$$H(\mu) = \frac{1}{\mu_1 \cdot \mu_2 \dots \mu_n} \frac{\prod_{i=1}^{n} (\mu + \mu_i)}{\prod_{\alpha=1}^{n} (1 + k_\alpha \mu)} \tag{27}$$

Therefore, using the Eq. (42) of Appendix B,

$$S(-\mu) = \left\{ (1 - \varpi_0) + \frac{1}{8} \lambda \varpi_0 (1 + \varpi_0) \right\}^{\frac{1}{2}} H(\mu) \tag{28}$$

Therefore, finally, using the Eq. (28), we get from the Eq. (22)

$$I(0, \mu) = \frac{3}{4} \left\{ (1 - \varpi_0) + \frac{1}{8} \lambda \varpi_0 (1 + \varpi_0) \right\}^{\frac{1}{2}} F H(\mu) \tag{29}$$

APPENDIX A: RELATION AMONG $D_\ell(x)$'s

Since

$$D_\ell(x) = \sum_j \frac{a_j \ \mu_j^\ell}{1 + \mu_j \ x}$$

$$\Rightarrow D_\ell(x) = \frac{1}{x} \left(\frac{2}{\ell} \epsilon_{\ \ell\ ,\ \text{odd}} - D_{\ell-1}(x) \right) \tag{30}$$

where

$$\epsilon_{\ \ell,\ \text{odd}} = \begin{cases} 1 & \text{if } \ell \text{ is odd} \\ 0 & \text{if } \ell \text{ is even} \end{cases} \tag{31}$$

But the Eq. (30) gives

$$D_{2j-1}(x) = \frac{1}{x} \left(\frac{2}{2j - 1} - D_{\ell-1}(x) \right) \tag{32}$$

$$D_{2j}(x) = -\frac{1}{x} D_{2j-1}(x) \tag{33}$$

From the two relations (32) and (33), we redily deduce that

$$D_{2j-1}(x) = \frac{2}{(2j-1)x} + \frac{2}{(2j-3)x^3} + \cdots + \frac{2}{3x^{2j-3}} + \frac{1}{x^{2j-1}}\{2 - D_0(x)\} \quad (j = 1, \ldots, 2n) \quad (34)$$

$$D_{2j}(x) = -\frac{2}{(2j-1)x^2} - \frac{2}{(2j-3)x^4} - \cdots - \frac{2}{3x^{2j-2}} - \frac{1}{x^{2j}}\{2 - D_0(x)\} \quad (j = 1, \ldots, 2n) \quad (35)$$

which are the Eqs. (24) and (25) of Chapter-III, pp. 73 of Chandrasekhar (1960)

Now from the Eq. (35), putting $j = 2$ and $j = 1$, we get

$$D_2(x) = -\frac{1}{x^2}\{2 - D_0(x)\}$$

and

$$D_4(x) = -\frac{2}{3x^2} - \frac{1}{x^4}\{2 - D_0(x)\}$$

i.e.

$$D_4(x)\left\{-\frac{1}{x^2}\{2 - D_0(x)\}\right\} = \left\{-\frac{2}{3x^2} - \frac{1}{x^4}\{2 - D_0(x)\}\right\} D_2(x)$$

$$D_2^2(x) - D_0(x) \cdot D_4(x) = \frac{2}{3} \times D_2(x) - 2 \cdot D_4(x)$$

Replacing x by k in the above equations and using the notation D_ℓ for $D_\ell(k)$, we get

$$D_0 D_4 - D_2^2 = 2D_4 - \frac{2}{3} D_2 \quad (36)$$

APPENDIX B: RELATION BETWEEN THE ROOTS OF THE CHARACTERISTIC EQUATION (15) AND THE ZEROS OF THE LEGENDRE POLYNOMIAL $P_{2n}(\mu)$:

Let p_{2j} be the co-efficients of μ^{2j} of the Legendre polynomial $P_{2n}(\mu)$. So,

$$P_{2n}(\mu) = \sum_{j=1}^{n} p_{2j}\, \mu^{2j} \quad (37)$$

Now, we consider

$$\sum_{j=1}^{n} p_{2j} D_{2j}(k) = \sum_{j} \frac{a_j}{1 + \mu_j\, k} \left(\sum_{j=1}^{n} p_{2j}\, \mu^{2j} \right) = 0$$

Since μ_i's are zeros of the Legendre polynomial $P_{2n}(\mu)$.
Therefore,

$$\sum_{j=1}^{n} p_{2j} D_{2j}(k) = 0 \tag{38}$$

The Eq. (38) is the required form of the Characteristic equation.
Now, we get

$$p_0 D_0 + \cdots + p_{2n} D_{2n} = 0$$

i.e.

$$\{2\lambda\varpi_0\,(2+\varpi_0) - 2\,(8+\lambda)\,\varpi_0 + 16\}\, p_{2n} t^n + \cdots + \{16\}\, p_0 = 0 \tag{39}$$

Therefore,

$$k_1^2 \times k_2^2 \ldots k_n^2 = (-1)^n \left\{(1-\varpi_0) + \frac{1}{8}\lambda\varpi_0(1+\varpi_0)\right\} \frac{p_{2n}}{p_0} \tag{40}$$

Again, μ_i's are zeros of the Legendre polynomial $P_{2n}(\mu)$ and so,

$$\mu_1^2 \times \mu_2^2 \cdots \mu_n^2 = (-1)^n \frac{p_0}{p_{2n}} \tag{41}$$

Multiplying the Eqs. (40) and (41), we get

$$k_1 \cdot k_2 \ldots k_n \cdot \mu_1 \cdot \mu_2 \ldots \mu_n = \left\{(1-\varpi_0) + \frac{1}{8}\lambda\varpi_0(1+\varpi_0)\right\}^{\frac{1}{2}} \tag{42}$$

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